SOLUTION OF A MIXED BOUNDARY-VALUE PROBLEM BY THE BUBNOV–GALERKIN METHOD

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A method is proposed of constructing a sequence of coordinate functions with the aid of R-functions [1-3], for the solution of a mixed boundary-value problem. The approximate solution to the given problem is qualitatively the same as that obtained by the relaxation method [6].

1. We consider the boundary-value problem

$$Lu = -\Delta u + cu = f,\ c < 0,\ (x,\ y) \in \Omega, \tag{1}$$

$$\left(\frac{\partial u}{\partial n} + \sigma_k u\right)\Big|_{\Gamma_1(k)} = 0 \qquad (k = 1,\ 2), \tag{2}$$

$$\frac{\partial u}{\partial n}\Big|_{\Gamma_2(k)} = 0. \tag{3}$$

Here Ω is a finite two-dimensional region bounded by a piecewise-smooth contour $\Gamma_0 = \sum_{i=1}^{2}\left(\Gamma_i^{(1)} \cup \Gamma_i^{(2)}\right)$, within which the inner normal $\nu_1^{(k)}$ (Fig. 1) is defined almost everywhere.

According to the Bubnov–Galerkin method, the approximate solution to the boundary-value problem (1)-(2) will be represented in the form

$$u_n(x,\ y) = \sum_{i=1}^{n} c_i \varphi_i(x,\ y), \tag{4}$$

where c_i are arbitrary constants yet to be determined and φ_i are coordinate functions, elements of the sequence

$$\varphi_1,\ \varphi_2,\ \ldots,\ \varphi_n,\ \ldots, \tag{5}$$

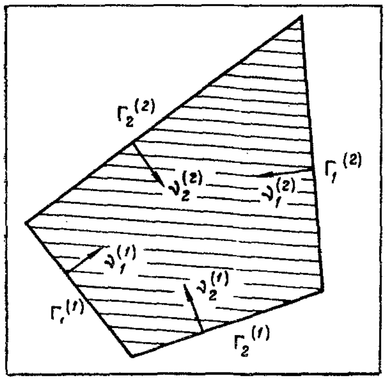

Fig. 1. General formulation of a mixed boundary-value problem.

which are a sufficient number of times continuously differentiable within the region Ω and which satisfy conditions (2)-(3) at its boundary Γ_0. The other requirements usually imposed on a system of coordinate functions (4) are also met, as will become immediately apparent after the system has been constructed by the following method.

We will show here the basic steps in constructing a sequence of coordinate functions.

By the procedures described in [1-3], we construct functions $w_s(x, y)$ which satisfy the conditions:

$$\omega_s|_{\Gamma_s} = 0, \tag{6}$$

$$\frac{\partial \omega_s}{\partial \nu_s}\Big|_{\Gamma_s} = 1, \tag{7}$$

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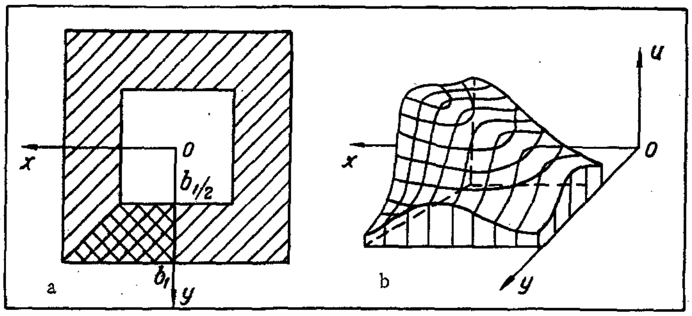

Fig. 2. Cross section of an electrical coil in the shape of a square frame (a) and surface representing the temperature distribution in a symmetrical coil element (b).

$$\omega_s > 0,\ (x,\ y) \in \Omega \qquad (s = 0,\ 1,\ 2). \tag{8}$$

We then introduce the linear differential operator D defined as follows:

$$Dv = a_0 v + a_1 \frac{\partial v}{\partial x} + a_2 \frac{\partial v}{\partial y}, \tag{9}$$

where

$$a_0 = \frac{\omega_2}{\omega_1 + \omega_2} \cdot \frac{\sigma_1 f_2 + \sigma_2 f_1}{f_1 + f_2}; \quad a_1 = -\omega'_{0x}; \quad a_2 = -\omega'_{0y}. \tag{10}$$

According to conditions (6)-(8), operator D has the following properties:

$$Dv|_{\Gamma_1^{(k)}} = \left(\frac{\partial v}{\partial n_k} + \sigma_k v\right)\Big|_{\Gamma_1^{(k)}}, \tag{11}$$

$$Dv|_{\Gamma_2^{(k)}} = 0. \tag{12}$$

The coordinate functions $\varphi_i(x, y)$ will be sought in the form

$$\varphi_i = \psi_{0i} + \omega_0 \psi_{1i}, \tag{13}$$

where ψ_{0i}, ψ_{1i} are arbitrary functions in the class $C^2(\Omega)$.

Let us consider the differential relation

$$Dv = \omega_0 g,\ g \in C^2(\Omega), \tag{14}$$

which is defined inside the region Ω and which at segments $\Gamma_1^{(k)}$, $\Gamma_2^{(k)}$ of the boundary becomes the respective condition (2).

Inserting expression (13) for φ_i into (14) yields

$$D(\psi_{0i}) + \psi_{1i} D\omega_0 + \omega_0 D\psi_{1i} = \omega_0 g. \tag{15}$$

In the vicinity of contour Γ_0

$$\omega_0|_{\Gamma_0} = 0(\omega_0),\ D\omega_0|_{\Gamma_0} = 1 + 0(\omega_0), \tag{16}$$

and, therefore, (15) becomes the condition

$$D\psi_{0i} + \psi_{1i} = \omega_0 g',\ g' \in C^2(\Omega). \tag{17}$$

From here

$$\psi_{1i} = -D\psi_{0i} + \omega_0 g'. \tag{18}$$

After inserting ψ_{1i} from (18) into (13), we have

$$\varphi_i = \psi_{0i} - \omega_0 D\psi_{0i} + \omega_0^2 g''. \tag{19}$$

Moreover, functions φ_i satisfy the boundary condition (2) exactly with any arbitrary functions ψ_{0i}, $g'' \in C^2(\Omega)$.

Next, let $\{\lambda_i\}_{i=1}^{\infty}$ denote a system of coordinate functions which is complete with respect to region Ω. Letting $g'' \equiv 0$ and $\psi_{0i} \equiv \lambda_i$ in (19), we obtain a final expression for the elements φ_i of the sought sequence of coordinate functions:

$$\varphi_i = \lambda_i - \omega_0 D\lambda_i. \tag{20}$$

The arbitrary constants c_i are determined from the system of equations

$$\sum_{i=1}^{n} (L\varphi_i, \varphi_j)\, c_i = (f, \varphi_j) \quad (j = 1, 2, \ldots, n). \tag{21}$$

2. As an example, we will solve the problem of determining the steady-state temperature field of an electrical coil wound in the shape of a square frame (Fig. 2a).

The thermal conductivity of the coil λ is assumed constant and Joule heat is generated in the coil according to the relation

$$q = q_0(1 + \alpha_0 u), \tag{22}$$

where q_0 denotes the quantity of heat generated at a fixed temperature u_0 and α_0 is the temperature coefficient. The outer surface s_1 and the inner surface s_2 of the coil transfer heat to the ambient medium according to Newton's law:

$$\left[\lambda \frac{\partial u}{\partial n_k} + \alpha_k (u - u_k)\right]_{s_k} = 0 \quad (k = 1, 2). \tag{23}$$

Here α_k is the heat transfer coefficient for the respective surface s_k, u_k is the ambient temperature at surface s_k, and n_k is the direction of the outer normal to s_k.

Assuming λ = 1.488 kcal/m · h · °C, q_0 = 768.96 kcal/h · m^3 · °C, $\alpha_2 = 4\alpha_1$, α_0 = 0.0036/°C, and u_0 = 0°C, we are now to determine the temperature field of the coil with the dimensions b_1 = 5.08 cm and b_2 = $1/2 b_1$.

The steady-state temperature distribution in a symmetrical section element of the coil (Fig. 2a) satisfies the differential equation

$$Lu = -\Delta u + cu = f, \quad f = \frac{q_0}{\lambda}, \quad c = -\alpha_0 f$$

with the boundary conditions

$$\left(\frac{\partial u}{\partial n} + \sigma_k u\right)\Big|_{\Gamma_1(k)} = h_k, \quad \sigma_k = \frac{\alpha_k}{\lambda}, \quad h_k = \sigma_k u_k,$$

$$\frac{\partial u}{\partial n}\Big|_{\Gamma_2(k)} = 0.$$

3. Under conditions of our problem, functions $w_s(x, y)$ become

$$\omega_1 = f_1 + f_2 - \sqrt{f_1^2 + f_2^2}, \tag{24}$$

$$\omega_2 = g_1 + g_2 - \sqrt{g_1^2 + g_2^2}, \tag{25}$$

$$\omega_0 = \omega_1 + \omega_2 - \sqrt{\omega_1^2 + \omega_2^2}, \tag{26}$$

where

$$f_1 = y - \frac{b_1}{2}; \quad f_2 = b_1 - y, \tag{27}$$

$$g_1 = y - x, \quad g_2 = x. \tag{28}$$

Using the properties of functions $w_s(x, y)$, we can easily construct function $\psi(x, y)$ satisfying conditions (2'). Function $\psi(x, y)$ will be expressed as

$$\Psi = -\frac{\omega_0\omega_2}{\omega_1 + \omega_2} \cdot \frac{h_1 f_2 + h_2 f_1}{f_1 + f_2}. \tag{29}$$

Let $\bar{u} = u - \psi$. Then function $\bar{u}$ satisfies the differential equation

$$L\bar{u} = -\Delta\bar{u} + c\bar{u} = f_1, \quad f_1 = \Delta\psi - c\psi + f$$

and the homogeneous boundary conditions

$$\left(\frac{\partial \bar{u}}{\partial n_k} + \sigma_k \bar{u}\right)\Big|_{\Gamma_1(k)} = 0,$$

$$\frac{\partial \bar{u}}{\partial n_k}\Big|_{\Gamma_2(k)} = 0.$$

Formula (4) represents an approximate solution to the boundary-value problem (1")-(3"). Under conditions of our problem, the coordinate functions $\varphi_i(x, y)$ are

$$\varphi_i = \lambda_i - \omega_0 D\lambda_i, \tag{30}$$

where

$$\{\lambda_i\}_1^\infty = \{1,\ x,\ y,\ x^2,\ xy,\ y^2,\ \ldots\}; \tag{31}$$

$$D\lambda_i = \frac{\omega_1}{\omega_1 + \omega_2} \cdot \frac{\sigma_1 f_2 + \sigma_2 f_1}{f_1 + f_2} \lambda_i - \left(\frac{\partial \lambda_i}{\partial x} \cdot \frac{\partial \omega_0}{\partial x} + \frac{\partial \lambda_i}{\partial y} \cdot \frac{\partial \omega_0}{\partial y}\right). \tag{32}$$

The arbitrary constants are determined from the system of Eqs. (20).

Calculations were made on a "Ural-2" computer for i = 6 and the following values were obtained for the coefficients: $c_1 = 147.8093$; $c_2 = -1202.2411$; $c_3 = -120.2417$; $c_4 = 25{,}701.951$; $c_5 = 22{,}203.284$, $c_6 = 15{,}701.951$.

The surface shown in Fig. 2b represents the steady-state temperature field of a symmetrical coil element.

This approximate solution (1) to the boundary-value problem (1')-(2') obtained by the Bubnov–Galerkin method is qualitatively the same as the solution obtained in [6] by the relaxation method.

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